

# The $N$ -soliton solution of the Degasperis-Procesi equation

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## Abstract

This paper extends the results of the previous paper designated I hereafter in which the one- and two-soliton solutions of the Degasperis-Procesi (DP) equation were obtained and their peakon limit was considered. Here, we present the general  $N$ -soliton solution of the DP equation and investigate its property. We show that it has a novel structure expressed by a parametric representation in terms of the BKP  $\tau$ -functions. A purely algebraic proof of the solution is given by establishing various identities among the  $\tau$ -functions. The large time asymptotic of the solution recovers the formula for the phase shift which was derived in I by a different method. Finally, the structure of the  $N$ -soliton solution is discussed in comparison with that of the Camassa-Holm shallow water wave equation.

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## 1. Introduction

The Degasperis-Procesi (DP) equation

$$u_t + 3\kappa^3 u_x - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx} \quad (1.1)$$

where  $u = u(x, t)$  and  $\kappa$  is a real constant, is a current research interest in soliton theory. Here, the subscripts  $t$  and  $x$  appended to  $u$  denote partial differentiation. This equation has been proposed as a candidate for the completely integrable nonlinear partial differential equations by using the method of asymptotic integrability [1]. A subsequent work has established its complete integrability by exhibiting some features common to the integrable system such as the existence of the Lax pair, infinity of conservation laws and so on [2]. Most studies so far have been devoted to the special case  $\kappa = 0$  in equation (1.1) [2-6]. Among them, a remarkable feature is that the corresponding equation exhibits peakon solutions which are represented by piecewise analytic functions. On the other hand, the case  $\kappa \neq 0$  merits a separate treatment and it was considered quite recently in our work, which is designated I hereafter [7]. Indeed, performing a reduction procedure for the soliton solutions of the Kadomtsev-Petviashvili (KP) hierarchy, we have obtained the explicit one- and two-soliton solutions and explored their properties. We have found that as opposed to the case  $\kappa = 0$ , solutions recover their analytic nature and behave like usual solitons. Nevertheless, several new features appear which are worth remarking. First, the soliton velocity depends nonlinearly on its amplitude as opposed to the usual linear relation. Second, the interaction process of two solitons reveals a new aspect which was never observed in the interaction of solitons for the typical soliton equation like the Korteweg-de Vries (KdV) equation. In particular, the small soliton exhibits a nonnegative phase shift for certain range of the amplitude parameters which is in striking contrast to the KdV case where the small soliton always exhibits a negative phase shift irrespective of values of the amplitude parameters.

The form of the two-soliton solution presented in I is very suggestive in constructing the general  $N$ -soliton solution where  $N$  is an arbitrary positive integer. We emphasize that it has a parametric representation in terms of two  $\tau$ -functions each of which has the standard Hirota form. This fact would enable us to surmise the functional form of the  $\tau$ -functions which produce the  $N$ -soliton solution. Nevertheless, the gap between the two-soliton and  $N$ -soliton cases is found to be so great that the construction of the  $N$ -soliton solution becomes a nontrivial problem which deserves further study.

The purpose of this paper is to provide the general  $N$ -soliton solution of the DP equation (1.1) in the form of the parametric representation. The solution presented here will be shown to exhibit several new features when compared with existing  $N$ -soliton solutions of typical nonlinear evolution equations like the KdV and KP equations. The procedure for constructing the  $N$ -soliton solution can be accomplished along the same

line as in the previous work I. However, we encounter several technical difficulties in the general  $N$ -soliton case which arise in reducing the proof of the solution to certain algebraic identities. Thus, the establishment of these identities turns out to be the main task in the present analysis. In conclusion, we show that as in the case of the two-soliton solution, the general  $N$ -soliton solution has a simple structure expressed by two fundamental  $\tau$ -functions. Remarkably, their functional form is found to be the same as the  $\tau$ -function for the  $N$ -soliton solution of a model shallow water wave equation introduced by Hirota and Satsuma [8]. In section 2, we summarize the results obtained in the previous study for constructing soliton solutions of the DP equation. In section 3, we carry out the proof of the  $N$ -soliton solution by means of a purely algebraic procedure in which the identities among the  $\tau$ -functions will play a central role. In section 4, we investigate the asymptotic behavior of the  $N$ -soliton solution for large time and recover the formula for the phase shift which has been derived in I by a different method. Section 5 is devoted to discussion where the structure of the  $N$ -soliton solution is compared with that of the Camassa-Holm (CH) shallow water wave equation. In appendix, we apply the CKP reduction to the  $\tau$ -function of the  $2N$ -soliton solution for the KP hierarchy and present the  $\tau$ -function which produces the  $N$ -soliton solution for a member of the CKP hierarchy.

## 2. Summary of the previous study

### 2.1. A system of equations equivalent to the DP equation

Here, we summarize the results associated with the DP equation while focusing on an equivalent system of equations to the DP equation and their soliton solutions. We describe only the main formulas without further explanation. For their derivation and implication, we refer to our work I as well as a related paper [2].

In order to solve equation (1.1) under the boundary condition  $u(\pm\infty, t) = 0$ , we introduce the new variable  $r$  by the relation

$$r^3 = u - u_{xx} + \kappa^3 \quad (2.1)$$

and recast equation (1.1) into the conservation form

$$r_t + (ru)_x = 0 \quad (2.2)$$

where the boundary condition for  $r$  is given by  $r(\pm\infty, t) = \kappa$ . A crucial step for simplifying the analysis is to define a coordinate transformation  $(x, t) \rightarrow (y, \tau)$  by

$$dy = rdx - rudt \quad (2.3a)$$

$$d\tau = dt. \quad (2.3b)$$

Consequently, the  $x$  and  $t$  derivatives are rewritten as

$$\frac{\partial}{\partial x} = r \frac{\partial}{\partial y}, \quad (2.4a)$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - ru \frac{\partial}{\partial y}. \quad (2.4b)$$

The transformation (2.3) is well-defined provided that  $r > 0$ . Under this condition, its inverse transformation is obtained by solving the system of equations for  $x$

$$x_y = \frac{1}{r(y, t)} \quad (2.5a)$$

$$x_t = u(y, t). \quad (2.5b)$$

Here in equation (2.5) and hereafter, we use the time variable  $t$  instead of  $\tau$  in view of (2.3b). Using the transformation (2.3), equations (2.1) and (2.2) become

$$u = -r(\ln r)_{ty} + r^3 - \kappa^3 \quad (2.6)$$

$$r_t + r^2 u_y = 0. \quad (2.7)$$

Note that by virtue of (2.7) the compatibility condition  $x_{yt} = x_{ty}$  is assured for the system of equations (2.5) that determine the inverse mapping.

## 2.2 Lax hierarchy of the CKP equation

To demonstrate the complete integrability of the system of equations (2.6) and (2.7), we substitute equation (2.6) into equation (2.7) to obtain

$$v_t = -\frac{3}{4}(r^2)_y \quad (2.8a)$$

where the new dependent variable  $v$  is defined by

$$v = -\frac{ryy}{2r} + \frac{r_y^2}{4r^2} - \frac{1}{4} \left( \frac{1}{r^2} - \frac{1}{\kappa^2} \right). \quad (2.8b)$$

Furthermore, we can eliminate the variable  $r$  from (2.8) and derive a single nonlinear equation for  $v$ . It reads

$$\begin{aligned} v_{tyyyy} + \left( 20v - \frac{5}{\kappa^2} \right) v_{tyy} + 30v_y v_{ty} + \left\{ 18v_{yy} + 16 \left( 2v - \frac{1}{2\kappa^2} \right)^2 \right\} v_t \\ + \left\{ 4v_{yyy} + 32 \left( 2v - \frac{1}{2\kappa^2} \right) v_y \right\} \left( \partial_y^{-1} v_t - \frac{3}{4} \kappa^2 \right) = 0 \end{aligned} \quad (2.9)$$

where  $\partial_y^{-1} = \int_{-\infty}^y dy$  is an integral operator. It has been pointed out that equation (2.9) with  $v + (1/4\kappa^2)$  in place of  $v$  is a member of the Lax hierarchy of the Kaup-Kuperschmidt (KK) (or CKP) equation. Thanks to this fact, we were able to construct explicitly the  $N$ -soliton solution of equation (2.9).

### 2.3. Procedure for constructing $N$ -soliton solution

The procedure for constructing the  $N$ -soliton solution of the DP equation consists of two steps, which we shall now demonstrate. First, we write the  $N$ -soliton solution of equation (2.9) in the form

$$v = \frac{3}{4}(\ln f)_{yy} \quad (2.10)$$

where  $f$  is obtained by means of a reduction for the  $2N$ -soliton solution of the KP hierarchy. See appendix for detail. It may be expressed in the form of finite sum

$$f = \sum_{\mu, \nu=0,1} \exp \left[ \sum_{i=1}^N (\mu_i + \nu_i) \xi_i + \sum_{i=1}^N (2\mu_i \nu_i - \mu_i - \nu_i) \ln a_i \right. \\ \left. + \frac{1}{2} \sum_{\substack{i,j=1 \\ (i \neq j)}}^N (\mu_i \mu_j + \nu_i \nu_j) A_{2i-1, 2j-1} + \frac{1}{2} \sum_{\substack{i,j=1 \\ (i \neq j)}}^N (\mu_i \nu_j + \mu_j \nu_i) A_{2i-1, 2j} \right] \quad (2.11a)$$

with

$$\xi_i = k_i \left( y - \frac{3\kappa^4}{1 - \kappa^2 k_i^2} t - y_{i0} \right), (i = 1, 2, \dots, N) \quad (2.11b)$$

$$a_i = \sqrt{\frac{1 - \frac{1}{4}\kappa^2 k_i^2}{1 - \kappa^2 k_i^2}}, (i = 1, 2, \dots, N) \quad (2.11c)$$

$$e^{A_{2i-1, 2j-1}} = \frac{(p_i - p_j)(q_i - q_j)}{(p_i + q_j)(q_i + p_j)}, (i, j = 1, 2, \dots, N; i \neq j) \quad (2.11d)$$

$$e^{A_{2i-1, 2j}} = \frac{(p_i - q_j)(q_i - p_j)}{(p_i + p_j)(q_i + q_j)}, (i, j = 1, 2, \dots, N) \quad (2.11e)$$

where the parameters  $p_i$  and  $q_i$  in (2.11d) and (2.11e) are given respectively by

$$p_i = \frac{k_i}{2} \left[ 1 + \frac{2}{\kappa k_i} \sqrt{\frac{1}{3} \left( 1 - \frac{1}{4}\kappa^2 k_i^2 \right)} \right], (i = 1, 2, \dots, N) \quad (2.11f)$$

$$q_i = \frac{k_i}{2} \left[ 1 - \frac{2}{\kappa k_i} \sqrt{\frac{1}{3} \left( 1 - \frac{1}{4}\kappa^2 k_i^2 \right)} \right], (i = 1, 2, \dots, N). \quad (2.11g)$$

Here,  $k_i$  and  $y_{i0}$  are the wavenumber and phase of the  $i$ th soliton, respectively and the conditions  $0 < \kappa k_i < 1, (i = 1, 2, \dots, N)$  are imposed to ensure the positivity of  $r$ . Thus,

the  $N$ -soliton solution of equation (2.9) is characterized completely by the  $2N$  parameters  $k_i$  and  $y_{i0}$  ( $i = 1, 2, \dots, N$ ). For  $N = 1, 2$ , the explicit expressions of  $f$  are written as

$$f = 1 + \frac{2}{a_1} e^{\xi_1} + e^{2\xi_1}, (N = 1) \quad (2.12a)$$

$$f = 1 + \frac{2}{a_1} e^{\xi_1} + \frac{2}{a_2} e^{\xi_2} + e^{2\xi_1} + e^{2\xi_2} + \frac{2\nu}{a_1 a_2} e^{\xi_1 + \xi_2} + \frac{2\delta}{a_2} e^{2\xi_1 + \xi_2} + \frac{2\delta}{a_1} e^{\xi_1 + 2\xi_2} + \delta^2 e^{2\xi_1 + 2\xi_2}, (N = 2) \quad (2.12b)$$

where

$$\delta = \frac{(k_1 - k_2)^2 [(k_1^2 - k_1 k_2 + k_2^2) \kappa^2 - 3]}{(k_1 + k_2)^2 [(k_1^2 + k_1 k_2 + k_2^2) \kappa^2 - 3]} \quad (2.12c)$$

$$\nu = \frac{(2k_1^4 - k_1^2 k_2^2 + 2k_2^4) \kappa^2 - 6(k_1^2 + k_2^2)}{(k_1 + k_2)^2 [(k_1^2 + k_1 k_2 + k_2^2) \kappa^2 - 3]}. \quad (2.12d)$$

Second, we substitute (2.10) into (2.8a) and integrate the resultant equation by  $y$  under the boundary condition  $r(\pm\infty, t) = \kappa$  and obtain the expression of  $r$  in terms of  $f$

$$r^2 = -(\ln f)_{ty} + \kappa^2. \quad (2.13)$$

To represent the solution in parametric form, we must integrate the system of equations (2.5). It follows from equation (2.5a) subject to the boundary condition  $r(\pm\infty, t) = \kappa$  that

$$x = \frac{y}{\kappa} + \int_{-\infty}^y \left( \frac{1}{r} - \frac{1}{\kappa} \right) dy + d \quad (2.14)$$

where  $d$  is an integration constant. This constant is independent of  $t$  as confirmed easily using equations (2.5b) and (2.7). Substituting  $r$  from (2.13) into (2.14) and performing the integration, we obtain the expression of the coordinate transformation  $x = x(y, t)$ , which, combined with (2.5b), gives a parametric representation of the  $N$ -soliton solution for the DP equation. In the simplest case of  $N = 1$ , the explicit form of the one-soliton solution reads

$$u(y, t) = \frac{\frac{8\kappa^3}{a_1} (a_1^2 - 1) (a_1^2 - \frac{1}{4})}{\cosh \xi_1 + 2a_1 - \frac{1}{a_1}} \quad (2.15a)$$

$$x(y, t) = \frac{y}{\kappa} + \ln \left( \frac{\alpha_1 + 1 + (\alpha_1 - 1) e^{\xi_1}}{\alpha_1 - 1 + (\alpha_1 + 1) e^{\xi_1}} \right). \quad (2.15b)$$

Here, the integration constant  $d$  has been set to  $d = \ln[(\alpha_1 + 1)/(\alpha_1 - 1)]$  with  $\alpha_1$  being expressed in terms of  $a_1$  as

$$\alpha_1 = \sqrt{\frac{(2a_1 - 1)(a_1 + 1)}{(2a_1 + 1)(a_1 - 1)}} \quad (2.15c)$$

and  $\xi_1$  and  $a_1$  are defined respectively by (2.11b) and (2.11c).

#### 2.4. Remark

The most difficult technical problem throughout the present analysis is to evaluate the integral in (2.14) with  $r$  constructed from the  $N$ -soliton  $\tau$ -function  $f$  through the relation (2.13). In I, this calculation has been performed only for the case of  $N = 1, 2$ . As will be developed in detail in the next section, the problem mentioned here is resolved completely by finding a suitable expression of  $r$  in terms of two fundamental  $\tau$ -functions. Another remark is concerned with the form of the  $\tau$ -function  $f$ . As shown in appendix, it can be represented by an equivalent determinantal form which has been used in I. However,  $f$  given by (2.11) is relevant in establishing the key identities (3.13) and (3.14) below since the  $\tau$ -functions  $g_1$  and  $g_2$  cannot be put into the form of determinants but they are expressed by pfaffians. Consequently, we can no longer rely on the various formulas for determinants used frequently in proving the analogous identities associated with the  $N$ -soliton solution of the CH equation [9].

### 3. $N$ -soliton solution

#### 3.1. Parametric representation

Now, the main result in this paper can be formulated as follows: The  $N$ -soliton solution of the DP equation (1.1) can be written compactly in a parametric representation

$$u(y, t) = \left( \ln \frac{g_1}{g_2} \right)_t \quad (3.1a)$$

$$x(y, t) = \frac{y}{\kappa} + \ln \frac{g_1}{g_2} + d. \quad (3.1b)$$

Here,  $g_1$  and  $g_2$  are given respectively by

$$g_1 = \sum_{\mu=0,1} \exp \left[ \sum_{i=1}^N \mu_i (\xi_i - \phi_i) + \sum_{\substack{i,j=1 \\ (i < j)}}^N \mu_i \mu_j \gamma_{ij} \right] \quad (3.2a)$$

$$g_2 = \sum_{\mu=0,1} \exp \left[ \sum_{i=1}^N \mu_i (\xi_i + \phi_i) + \sum_{\substack{i,j=1 \\ (i < j)}}^N \mu_i \mu_j \gamma_{ij} \right] \quad (3.2b)$$

with

$$e^{-\phi_i} = \sqrt{\frac{(1 - \frac{\kappa k_i}{2})(1 - \kappa k_i)}{(1 + \frac{\kappa k_i}{2})(1 + \kappa k_i)}} \quad (i = 1, 2, \dots, N) \quad (3.2c)$$

$$\begin{aligned} e^{\gamma_{ij}} &= \frac{(p_i - p_j)(p_i - q_j)(q_i - p_j)(q_i - q_j)}{(p_i + p_j)(p_i + q_j)(q_i + p_j)(q_i + q_j)} \\ &= \frac{(k_i - k_j)^2 [(k_i^2 - k_i k_j + k_j^2) \kappa^2 - 3]}{(k_i + k_j)^2 [(k_i^2 + k_i k_j + k_j^2) \kappa^2 - 3]}, \quad (i, j = 1, 2, \dots, N; i \neq j) \end{aligned} \quad (3.2d)$$

where the phase variable  $\xi_i$  of the  $i$ th soliton is defined by (2.11b). In deriving the second line of (3.2d), we have used (2.11f) and (2.11g). Note remarkably that  $g_1$  and  $g_2$  have the same functional form except for the phase factors  $\pm\phi_i$ .

### 3.2. Identities among $\tau$ -functions

We verify (3.1) in a sequence of steps. To this end, we first define the following quantities constructed from the  $\tau$ -function introduced in (A.1):

$$\tilde{f} = \tau_{2N}(\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_{2N}) \quad (3.3)$$

$$\tilde{f}_1 = \tau_{2N}(\tilde{\xi}_1 + \ln(\tilde{p}_1/\tilde{q}_1), \tilde{\xi}_2 + \ln(\tilde{p}_2/\tilde{q}_2), \dots, \tilde{\xi}_{2N} + \ln(\tilde{p}_{2N}/\tilde{q}_{2N})) \quad (3.4)$$

$$\tilde{f}_2 = \tau_{2N}(\tilde{\xi}_1 + \ln(\tilde{q}_1/\tilde{p}_1), \tilde{\xi}_2 + \ln(\tilde{q}_2/\tilde{p}_2), \dots, \tilde{\xi}_{2N} + \ln(\tilde{q}_{2N}/\tilde{p}_{2N})). \quad (3.5)$$

An important observation here is that the following bilinear identity holds among  $\tilde{f}$ ,  $\tilde{f}_1$  and  $\tilde{f}_2$ :

$$\kappa^2 \tilde{f}^2 - \tilde{f} \tilde{f}_{ty} + \tilde{f}_t \tilde{f}_y = \kappa^2 \tilde{f}_1 \tilde{f}_2. \quad (3.6)$$

We have already met with this type of identity in a previous paper dealing with the  $N$ -soliton solution of the CH equation [9]. To modify (3.6) into a known identity, it is sufficient to replace the expressions of  $\tilde{f}$ ,  $\tilde{f}_1$  and  $\tilde{f}_2$  given above by their determinantal forms using the relation [10]

$$\sum_{\mu=0,1} \exp \left[ \sum_{i=1}^{2N} \mu_i \tilde{\xi}_i + \sum_{\substack{i,j=1 \\ (i < j)}}^{2N} \mu_i \mu_j \tilde{A}_{ij} \right] = \lambda_{2N} \det \left( \frac{e^{\tilde{\xi}_i} \delta_{ij}}{\tilde{p}_i - \tilde{q}_i} + \frac{1}{\tilde{p}_i - \tilde{q}_j} \right)_{1 \leq i,j \leq 2N} \quad (3.7a)$$

where

$$\tilde{\xi}_i = \tilde{\xi}_i + \sum_{\substack{j=1 \\ (j \neq i)}}^{2N} \tilde{A}_{ij}, \quad (i = 1, 2, \dots, 2N) \quad (3.7b)$$

$\lambda_{2N}$  is a constant factor given by

$$\lambda_{2N} = \exp \left[ - \sum_{\substack{i,j=1 \\ (i < j)}}^{2N} \tilde{A}_{ij} \right] \prod_{i=1}^{2N} (\tilde{p}_i - \tilde{q}_i) \quad (3.7c)$$

and  $\delta_{ij}$  is Kronecker's delta. Then, we can confirm that (3.6) reduces to the bilinear identity (23) of [9].

Let us now apply the CKP reduction defined by (A.4) to  $\tilde{f}$ ,  $\tilde{f}_1$  and  $\tilde{f}_2$  and subsequently shift the phase variables as  $\xi_i \rightarrow \xi_i - \ln a_i$  ( $i = 1, 2, \dots, N$ ) to introduce the new quantities  $f$  and  $g$

$$f = \tilde{f}|_{CKP}(\xi_i \rightarrow \xi_i - \ln a_i) \quad (3.8)$$



$$g = \tilde{f}_1|_{CKP}(\xi_i \rightarrow \xi_i - \ln a_i) = \tilde{f}_2|_{CKP}(\xi_i \rightarrow \xi_i - \ln a_i). \quad (3.9)$$

Here,  $g$  is represented by the finite sum analogous to the expression of  $f$

$$g = \sum_{\mu, \nu=0,1} \exp \left[ \sum_{i=1}^N (\mu_i + \nu_i) \xi_i + \sum_{i=1}^N (2\mu_i \nu_i - \mu_i - \nu_i) \ln a_i + \sum_{i=1}^N (\mu_i - \nu_i) \ln \left( -\frac{q_i}{p_i} \right) \right. \\ \left. + \frac{1}{2} \sum_{\substack{i,j=1 \\ (i \neq j)}}^N (\mu_i \mu_j + \nu_i \nu_j) A_{2i-1, 2j-1} + \frac{1}{2} \sum_{\substack{i,j=1 \\ (i \neq j)}}^N (\mu_i \nu_j + \mu_j \nu_i) A_{2i-1, 2j} \right]. \quad (3.10)$$

Note that only the difference between  $f$  and  $g$  is the third term on the right-hand side of (3.10). It is easy to see that both  $\tilde{f}_1$  and  $\tilde{f}_2$  reduce to the expression of  $g$  given above as a consequence of the CKP reduction.

It turns out from (3.6), (3.8) and (3.9) that

$$\kappa^2 - (\ln f)_{ty} = \kappa^2 \left( \frac{g}{f} \right)^2. \quad (3.11)$$

Comparing (3.11) with (2.13) and taking account of the boundary condition  $r(\pm\infty, t) = \kappa$ , we obtain the simplified expression of  $r$  in terms of  $f$  and  $g$

$$r = \kappa \frac{g}{f}. \quad (3.12)$$

Thus,  $r$  becomes a rational function of  $e^{\xi_i}$  ( $i = 1, 2, \dots, N$ ). This significant expression would not be derived without noticing the identity (3.6) since  $r$  from (2.13) leads simply to an irrational function of  $e^{\xi_i}$  ( $i = 1, 2, \dots, N$ ). The compact expression (3.12) enables us to integrate (2.14) explicitly when coupled with the following set of identities among  $f, g, g_1$  and  $g_2$ :

$$f = g_1 g_2 + \kappa(g_{1,y} g_2 - g_1 g_{2,y}) \quad (3.13)$$

$$g = g_1 g_2. \quad (3.14)$$

The proof of the above identities will be done by mathematical induction similar to that used successfully for the proof of the  $N$ -soliton solutions of various soliton equations within the framework of the bilinear formalism [11].

### 3.2.1. Proof of (3.13)

The identity (3.13) is established by comparing the coefficient of the factor  $\exp \left[ \sum_{i=1}^n \xi_i + \sum_{i=n+1}^m 2\xi_i \right]$  ( $0 \leq n < m \leq N$ ) on both sides. Let  $L_{m,n}$  be the corresponding coefficient of  $f$  and  $R_{m,n}$  be the one for the right-hand side of (3.13). Correspondingly, the summation with respect to  $\mu_i$  and  $\nu_i$  must be performed under the conditions

$$\mu_i + \nu_i = 1, (i = 1, 2, \dots, n), \quad \mu_i = \nu_i = 1, (i = n+1, n+2, \dots, m),$$

$$\mu_i = \nu_i = 0, (i = m + 1, m + 2, \dots, N). \quad (3.15)$$

For the purpose of performing the proof effectively, it is convenient to introduce the new variables  $\sigma_i$  according to the relations

$$\mu_i = \frac{1}{2}(1 + \sigma_i), \quad \nu_i = \frac{1}{2}(1 - \sigma_i), (i = 1, 2, \dots, N) \quad (3.16)$$

where  $\sigma_i$  takes either the value  $+1$  or  $-1$ . Consequently

$$\mu_i \mu_j + \nu_i \nu_j = \frac{1}{2}(1 + \sigma_i \sigma_j), (i, j = 1, 2, \dots, N) \quad (3.17a)$$

$$\mu_i \nu_j + \mu_i \nu_i = \frac{1}{2}(1 - \sigma_i \sigma_j), (i, j = 1, 2, \dots, N). \quad (3.17b)$$

Taking account of the relations which follow from (2.11d)-(2.11g) and (3.2d)

$$A_{2i-1, 2i} = 2 \ln a_i, (i = 1, 2, \dots, N) \quad (3.18a)$$

$$A_{2i-1, 2j} + A_{2i, 2j-1} = \gamma_{ij}, (i, j = 1, 2, \dots, N; i \neq j) \quad (3.18b)$$

as well as (3.15) and (3.17), we find that

$$\begin{aligned} L_{m,n} = \sum_{\sigma=\pm 1} \exp \left[ - \sum_{i=1}^n \ln a_i + \frac{1}{4} \sum_{\substack{i,j=1 \\ (i \neq j)}}^n (1 + \sigma_i \sigma_j) \gamma_{ij} \right. \\ \left. - \frac{1}{2} \sum_{\substack{i,j=1 \\ (i \neq j)}}^n \sigma_i \sigma_j A_{2i-1, 2j} + \sum_{i=n+1}^m \sum_{\substack{j=1 \\ (j \neq i)}}^m \gamma_{ij} \right] \quad (3.19) \\ R_{m,n} = \sum_{\sigma=\pm 1} \left( 1 + \kappa \sum_{i=1}^n \sigma_i k_i \right) \exp \left[ - \sum_{i=1}^n \sigma_i \phi_i + \frac{1}{4} \sum_{\substack{i,j=1 \\ (i \neq j)}}^n (1 + \sigma_i \sigma_j) \gamma_{ij} + \sum_{i=n+1}^m \sum_{\substack{j=1 \\ (j \neq i)}}^m \gamma_{ij} \right]. \quad (3.20) \end{aligned}$$

To simplify (3.19) further, we use the relation

$$\begin{aligned} \exp \left[ \frac{1}{2} \{ \gamma_{ij} + \sigma_i \sigma_j (A_{2i-1, 2j-1} - A_{2i-1, 2j}) \} \right] \\ = \frac{1}{2} (1 + \sigma_i \sigma_j) \frac{(p_i - p_j)(q_i - q_j)}{(p_i + q_j)(q_i + p_j)} + \frac{1}{2} (1 - \sigma_i \sigma_j) \frac{(p_i - q_j)(q_i - p_j)}{(p_i + p_j)(q_i + q_j)} \\ = \frac{1}{2} \frac{b_{ij}}{d_{ij}} + \frac{1}{2} \sigma_i \sigma_j \frac{\beta_i \beta_j}{d_{ij}}, (i, j = 1, 2, \dots, N; i \neq j) \quad (3.21a) \end{aligned}$$

where

$$\beta_i = 2\sqrt{3}k_i \sqrt{1 - \frac{1}{4}\kappa^2 k_i^2}, (i = 1, 2, \dots, N) \quad (3.21b)$$

$$b_{ij} = (2k_i^4 - k_i^2 k_j^2 + 2k_j^4)\kappa^2 - 6(k_i^2 + k_j^2), (i, j = 1, 2, \dots, N) \quad (3.21c)$$

$$d_{ij} = (k_i + k_j)^2 [(k_i^2 + k_i k_j + k_j^2)\kappa^2 - 3], (i, j = 1, 2, \dots, N). \quad (3.21d)$$

The first expression of (3.21a) is a simple consequence of (2.11d), (2.11e) and (3.2d) while the second expression follows by substituting (2.11f) and (2.11g) into the first one. Thus,  $L_{m,n}$  becomes

$$L_{m,n} = c_{m,n} \sum_{\sigma=\pm 1} \prod_{\substack{i,j=1 \\ (i < j)}}^n \left[ \frac{1}{2} (b_{ij} + \sigma_i \sigma_j \beta_i \beta_j) \right] \quad (3.22a)$$

where  $c_{m,n}$  is a factor independent of  $\sigma_i (i = 1, 2, \dots, n)$  given by

$$c_{m,n} = \frac{\exp \left[ \sum_{i=n+1}^m \sum_{\substack{j=1 \\ (j \neq i)}}^m \gamma_{ij} \right]}{\prod_{i=1}^n a_i \prod_{\substack{i,j=1 \\ (i < j)}}^n d_{ij}}. \quad (3.22b)$$

Similarly, invoking the relations

$$e^{-\sigma_i \phi_i} = \frac{1}{a_i} \frac{1 - \frac{\kappa}{2} \sigma_i k_i}{1 + \kappa \sigma_i k_i}, (i = 1, 2, \dots, N) \quad (3.23a)$$

$$\exp \left[ \frac{1}{2} (1 + \sigma_i \sigma_j) \gamma_{ij} \right] = \frac{(\sigma_i k_i - \sigma_j k_j)^2 [(k_i^2 - \sigma_i \sigma_j k_i k_j + k_j^2)\kappa^2 - 3]}{(k_i + k_j)^2 [(k_i^2 + k_i k_j + k_j^2)\kappa^2 - 3]},$$

$$(i, j = 1, 2, \dots, N; i \neq j) \quad (3.23b)$$

which follow from (2.11c), (3.2c) and (3.2d), we can modify  $R_{m,n}$  in the form

$$R_{m,n} = c_{m,n} \sum_{\sigma=\pm 1} (1 + \kappa \sum_{i=1}^n \sigma_i k_i) \prod_{i=1}^n \frac{1 - \frac{\kappa}{2} \sigma_i k_i}{1 + \kappa \sigma_i k_i}$$

$$\times \prod_{\substack{i,j=1 \\ (i < j)}}^n (\sigma_i k_i - \sigma_j k_j)^2 [(k_i^2 - \sigma_i \sigma_j k_i k_j + k_j^2)\kappa^2 - 3]. \quad (3.24)$$

Define  $P_n$  by the relation

$$R_{m,n} - L_{m,n} = c_{m,n} P_n(k_1, k_2, \dots, k_n) \quad (3.25a)$$

where

$$P_n(k_1, k_2, \dots, k_n) = \sum_{\sigma=\pm 1} \left( 1 + \kappa \sum_{i=1}^n \sigma_i k_i \right) \prod_{i=1}^n \frac{1 - \frac{\kappa}{2} \sigma_i k_i}{1 + \kappa \sigma_i k_i}$$

$$\begin{aligned}
& \times \prod_{\substack{i,j=1 \\ (i < j)}}^n (\sigma_i k_i - \sigma_j k_j)^2 [(k_i^2 - \sigma_i \sigma_j k_i k_j + k_j^2) \kappa^2 - 3] \\
& - \sum_{\sigma=\pm 1} \prod_{\substack{i,j=1 \\ (i < j)}}^n \left[ \frac{1}{2} (b_{ij} + \sigma_i \sigma_j \beta_i \beta_j) \right]. \tag{3.25b}
\end{aligned}$$

To proceed, we make an important remark. Let  $P_{1,n}$  be the first term on the right-hand side of (3.25b) multiplied by a factor  $\prod_{i=1}^n (1 - \kappa^2 k_i^2)$ . By virtue of the summation with respect to  $\sigma_i (i = 1, 2, \dots, n)$ , we can see that  $P_{1,n}$  is symmetric and even function of  $k_i (i = 1, 2, \dots, n)$  and  $P_{1,n}|_{k_1=\pm 1/\kappa} = 0$ . Hence,  $P_{1,n}$  has a factor  $\prod_{i=1}^n (k_i^2 - 1/\kappa^2)$  in view of symmetry. This implies that the first term is not a rational function but a polynomial. Furthermore, since the second term on the right-hand side of (3.25b) is an even function of  $\beta_i (i = 1, 2, \dots, n)$  due to the summation with respect to  $\sigma_i$ , only the even power of  $\beta_i$  contributes to the summation. This fact indicates that the second term is indeed a polynomial of  $k_i (i = 1, 2, \dots, n)$ . As a consequence,  $P_n$  becomes a polynomial of  $k_i (i = 1, 2, \dots, n)$ .

With the above remark in mind, we now prove the identity  $P_n = 0 (n = 1, 2, \dots, N)$  by mathematical induction. A direct calculation shows that  $P_1 = P_2 = 0$ . We assume that  $P_{n-2} = P_{n-1} = 0$ . Then

$$P_n|_{k_1=0} = 2 \prod_{i=2}^n k_i^2 (\kappa^2 k_i^2 - 3) P_{n-1}(k_2, k_3, \dots, k_n) \tag{3.26a}$$

$$P_n|_{k_1=\pm 2/\kappa} = 2 \prod_{i=2}^n [(\kappa^2 k_i^2 - 1)(\kappa^2 k_i^2 - 4)/\kappa^2] P_{n-1}(k_2, k_3, \dots, k_n) \tag{3.26b}$$

$$\begin{aligned}
P_n|_{k_1=k_2} &= 6k_1^2 (\kappa^2 k_1^2 - 4) \prod_{i=3}^n (k_i^2 - k_1^2)^2 \{ (k_i^4 + k_1^2 k_i^2 + k_1^4) \kappa^4 - 6(k_i^2 + k_1^2) \kappa^2 + 9 \} \\
&\quad \times P_{n-2}(k_3, k_3, \dots, k_n). \tag{3.26c}
\end{aligned}$$

On account of the properties (3.26) as well as symmetry and evenness, we see that  $P_n$  can be factored by a polynomial

$$\prod_{i=1}^n k_i^2 \left( k_i^2 - \frac{4}{\kappa^2} \right) \prod_{\substack{i,j=1 \\ (i < j)}}^n (k_i^2 - k_j^2)^2$$

of  $k_i (i = 1, 2, \dots, n)$  of degree  $2n^2 + 2n$ . On the other hand, it is obvious from (3.25b) that  $P_n$  is a polynomial of  $k_i (i = 1, 2, \dots, n)$  of degree  $2n(n-1) + 1$  at most, which is impossible except for  $P_n \equiv 0$ . This completes the proof.

### 3.2.2. Proof of (3.14)

The proof of the identity (3.14) can be done in the same way. By comparing the factor  $\exp[\sum_{i=1}^n \xi_i + \sum_{i=n+1}^m 2\xi_i]$  ( $0 \leq n < m \leq N$ ) on both sides of (3.14), we obtain

$$\begin{aligned} & \sum_{\sigma=\pm 1} \exp \left[ \sum_{i=1}^n \sigma_i \ln \left( -\frac{q_i}{p_i} \right) - \sum_{i=1}^n \ln a_i + \frac{1}{4} \sum_{\substack{i,j=1 \\ (i \neq j)}}^n (1 + \sigma_i \sigma_j) \gamma_{ij} - \frac{1}{2} \sum_{\substack{i,j=1 \\ (i \neq j)}}^n \sigma_i \sigma_j A_{2i-1,2j} \right. \\ & \left. + \sum_{i=n+1}^m \sum_{\substack{j=1 \\ (j \neq i)}}^m \gamma_{ij} \right] = \sum_{\sigma=\pm 1} \exp \left[ - \sum_{i=1}^n \sigma_i \phi_i + \frac{1}{4} \sum_{\substack{i,j=1 \\ (i \neq j)}}^n (1 + \sigma_i \sigma_j) \gamma_{ij} + \sum_{i=n+1}^m \sum_{\substack{j=1 \\ (j \neq i)}}^m \gamma_{ij} \right]. \quad (3.27) \end{aligned}$$

The calculation leading to (3.25) is applied as well to modify (3.27) further. With the aid of the relation

$$\left( -\frac{q_i}{p_i} \right)^{\sigma_i} = \frac{1 + \frac{1}{2} \kappa^2 k_i^2 - \frac{\kappa}{2} \sigma_i \beta_i}{1 - \kappa^2 k_i^2}, \quad (i = 1, 2, \dots, N) \quad (3.28)$$

which is derived simply from (2.11f) and (2.11g), we can recast (3.27) into the form

$$\begin{aligned} Q_n(k_1, k_2, \dots, k_n) & \equiv \sum_{\sigma=\pm 1} \prod_{i=1}^n \left( 1 + \frac{1}{2} \kappa^2 k_i^2 - \frac{3}{2} \kappa \sigma_i k_i \right) \\ & \times \prod_{\substack{i,j=1 \\ (i < j)}}^n (\sigma_i k_i - \sigma_j k_j)^2 [(k_i^2 - \sigma_i \sigma_j k_i k_j + k_j^2) \kappa^2 - 3] \\ & - \sum_{\sigma=\pm 1} \prod_{i=1}^n \left( 1 + \frac{1}{2} \kappa^2 k_i^2 - \frac{1}{2} \kappa \sigma_i \beta_i \right) \prod_{\substack{i,j=1 \\ (i < j)}}^n \left[ \frac{1}{2} (b_{ij} + \sigma_i \sigma_j \beta_i \beta_j) \right] = 0. \quad (3.29) \end{aligned}$$

Note that the second term on middle line in (3.29) is an even function of  $\beta_i$  ( $i = 1, 2, \dots, n$ ) so that  $Q_n$  becomes a polynomial of  $k_i$  ( $i = 1, 2, \dots, n$ ). We now prove the identity (3.29) by mathematical induction. The identity holds for  $n = 1, 2$ , as checked easily. We assume that  $Q_{n-2} = Q_{n-1} = 0$ . Then

$$Q_n|_{k_1=0} = 2 \prod_{i=2}^n k_i^2 (\kappa^2 k_i^2 - 3) Q_{n-1}(k_2, k_3, \dots, k_n) \quad (3.30a)$$

$$Q_n|_{k_1=\pm 2/\kappa} = 6 \prod_{i=2}^n [(\kappa^2 k_i^2 - 1)(\kappa^2 k_i^2 - 4)/\kappa^2] Q_{n-1}(k_2, k_3, \dots, k_n) \quad (3.30b)$$

$$Q_n|_{k_1=k_2} = 6k_1^2 (\kappa^2 k_1^2 - 4) (\kappa^2 k_1^2 - 1)^2 \prod_{i=3}^n (k_i^2 - k_1^2)^2 \{ (k_i^4 + k_1^2 k_i^2 + k_1^4) \kappa^4 - 6(k_i^2 + k_1^2) \kappa^2 + 9 \}$$

$$\times Q_{n-2}(k_3, k_3, \dots, k_n). \quad (3.30c)$$

The symmetry and evenness of  $Q_n$  with respect to  $k_i (i = 1, 2, \dots, n)$  as well as the properties (3.30) imply that  $Q_n$  has a factor

$$\prod_{i=1}^n k_i^2 \left( k_i^2 - \frac{4}{\kappa^2} \right) \prod_{\substack{i,j=1 \\ (i < j)}}^n (k_i^2 - k_j^2)^2$$

whose degree is  $2n^2 + 2n$ . But, as seen from (3.29), the degree of  $Q_n$  is  $2n^2$  at most and hence  $Q_n$  must vanish identically, completing the proof.

### 3.3. Proof of the $N$ -soliton solution

Once the identities (3.13) and (3.14) have been established, the proof of the  $N$ -soliton solution can be done straightforwardly. Indeed, if we substitute (3.12) with (3.13) and (3.14) into (2.14) and integrate it with respect to  $y$ , we obtain (3.1b). The expression (3.1a) follows immediately from (3.1b) by a simple differentiation with respect to  $t$  while using (2.5b) and the constancy of  $d$ .

### 3.4. Remark

The structure of the  $N$ -soliton solution thus obtained is worth elucidating. The present analysis shows that the  $\tau$ -functions  $g_1$  and  $g_2$  defined by (3.2) characterize completely the  $N$ -soliton solution of the DP equation. Another  $\tau$ -function  $f$  introduced in (2.11) is then expressed in terms of  $g_1$  and  $g_2$  as indicated by the key identity (3.13). It is interesting to recall that  $g_1$  and  $g_2$  have the same structure as the  $\tau$ -function for the  $N$ -soliton solution of a model equation for shallow water waves introduced by Hirota and Satsuma [8] which is a member of the BKP hierarchy in view of the classification theory of soliton equations [12, 13]. The identity (3.13) shows that the CKP  $\tau$ -function  $f$  can be represented by the BKP  $\tau$ -functions  $g_1$  and  $g_2$ .

Although we have addressed equation (1.1) with a nonzero  $\kappa$ , the case  $\kappa = 0$  is an interesting problem to call a special attention. As already mentioned in introduction, the search dealing with this issue has already been done by several authors [2-6]. In particular, an important problem associated with nonanalytic peakon solutions [3, 5] is how to reduce them from analytic soliton solutions. By taking a limit  $\kappa \rightarrow 0$ , a single peakon solution has been reproduced from the one-soliton solution. However, the general  $N$ -peakon case is still left as an open problem even though we have a numerical evidence showing that the two-soliton solution with very small  $\kappa$  approximates quite well the two-peakon solution. See section 4.3 of I.

Another limit  $\kappa \rightarrow \infty$  will also deserve remarking. It has been shown that if one replaces the velocity of the  $i$ th soliton by  $k_i^4$  in (2.11b) and takes the limit  $\kappa \rightarrow \infty$ , then the  $\tau$ -function reduced from  $f$  given by (2.11a) provides the  $N$ -soliton solution of the following fifth-order KdV equation introduced by Kaup [14] which belongs to a member

of the CKP hierarchy [13, 15]

$$v_t + \left( v_{yyyy} + 20vv_{yy} + \frac{80}{3}v^3 + 15v_y^2 \right)_y = 0. \quad (3.31)$$

To clarify the structure of the limiting form of the  $\tau$ -function  $f$ , it is suitable to use (3.2) and (3.13). We find in the limit of  $\kappa \rightarrow \infty$  that  $g_1$  and  $g_2$  are expanded in inverse power of  $\kappa$  as

$$g_1 \sim h - \frac{1}{\kappa}h_{t'} + O\left(\frac{1}{\kappa^2}\right) \quad (3.32a)$$

$$g_2 \sim h + \frac{1}{\kappa}h_{t'} + O\left(\frac{1}{\kappa^2}\right) \quad (3.32b)$$

where

$$h = \sum_{\mu=0,1} \exp \left[ \sum_{i=1}^N \mu_i \hat{\xi}_i + \sum_{\substack{i,j=1 \\ (i < j)}}^N \mu_i \mu_j \hat{\gamma}_{ij} \right] \quad (3.33a)$$

$$\hat{\xi}_i = k_i \left( y - k_i^4 t + \frac{3}{k_i^2} t' - y_{i0} \right), (i = 1, 2, \dots, N) \quad (3.33b)$$

$$e^{\hat{\gamma}_{ij}} = \frac{(k_i - k_j)^2 (k_i^2 - k_i k_j + k_j^2)}{(k_i + k_j)^2 (k_i^2 + k_i k_j + k_j^2)}, (i, j = 1, 2, \dots, N; i \neq j). \quad (3.33c)$$

Here,  $t'$  is an auxiliary time variable which may be either set to zero or absorbed into the phase constant  $y_{i0}$  in the final stage of the calculation. Substitution of (3.32) into (3.13) yields

$$f \sim \hat{f} \equiv h^2 - 2(hh_{t'y} - h_{t'}h_y) \quad (3.34)$$

and  $v$  with  $f = \hat{f}$  in (2.10) gives the  $N$ -soliton solution of equation (3.31). Hence, a single  $\tau$ -function  $h$  characterizes completely the  $N$ -soliton structure. See also [16-18] as for the construction of soliton solutions to equation (3.31) by means of the bilinear approach.

#### 4. Asymptotic behaviour of the $N$ -soliton solution

The asymptotic form of the  $N$ -soliton solution has been investigated in I. Here, we provide an alternative but more straightforward way to derive it on the basis of the asymptotic form of the  $\tau$ -functions  $g_1$  and  $g_2$ . To proceed, we order the magnitude of the soliton velocity in the  $(x, t)$  coordinate system as

$$c_1 > c_2 > \dots > c_N \quad (4.1a)$$

with

$$c_i = \frac{3\kappa^3}{1 - \kappa^2 k_i^2}, (i = 1, 2, \dots, N) \quad (4.1b)$$

and then transform to a moving reference frame with a constant velocity  $c_i$ . We first take the limit  $t \rightarrow -\infty$  with the phase variable  $\xi_i$  of the  $i$ th soliton being fixed. Since other phase variables tend to  $\pm\infty$  as

$$\xi_1, \xi_2, \dots, \xi_{i-1} \rightarrow +\infty, \quad \xi_{i+1}, \xi_{i+2}, \dots, \xi_N \rightarrow -\infty \quad (4.2)$$

the  $\tau$ -functions  $g_1$  and  $g_2$  given respectively by (3.2a) and (3.2b) are found to have the leading-order asymptotics

$$g_1 \sim \exp \left[ \sum_{j=1}^{i-1} (\xi_j - \phi_j) \right] \left( 1 + e^{\xi_i - \phi_i + \gamma_i^{(+)}} \right) \quad (4.3a)$$

$$g_2 \sim \exp \left[ \sum_{j=1}^{i-1} (\xi_j + \phi_j) \right] \left( 1 + e^{\xi_i + \phi_i + \gamma_i^{(+)}} \right) \quad (4.3b)$$

where

$$\gamma_i^{(+)} = \sum_{j=1}^{i-1} \gamma_{ij}, \quad (i = 1, 2, \dots, N). \quad (4.3c)$$

The asymptotic forms of  $u$  and  $x$  follow simply by substituting (4.3) into (3.1). They can be written as

$$u \sim u_i(\xi_i + \gamma_i^{(+)}) \quad (4.4a)$$

$$x - c_i t - x_{i0} \sim \frac{\xi_i}{\kappa k_i} + \ln \left( \frac{1 + e^{\xi_i - \phi_i + \gamma_i^{(+)}}}{1 + e^{\xi_i + \phi_i + \gamma_i^{(+)}}} \right) - 2 \sum_{j=1}^{i-1} \phi_j + d \quad (4.4b)$$

where  $u_i(\xi_i)$  is the one-soliton solution

$$u_i(\xi_i) = \frac{\kappa k_i c_i \sinh \phi_i}{\cosh \xi_i + \cosh \phi_i} \quad (4.4c)$$

and  $x_{i0} = y_{i0}/\kappa$ . We can confirm that the above expression of  $u_i$  coincides with the one-soliton solution given by (2.15a). Actually, by virtue of the relations

$$\cosh \phi_i = 2a_i - \frac{1}{a_i} \quad (4.5a)$$

$$\sinh \phi_i = \frac{1}{a_i} \sqrt{(4a_i^2 - 1)(a_i^2 - 1)} \quad (4.5b)$$

$$\kappa k_i = \sqrt{\frac{a_i^2 - 1}{a_i^2 - \frac{1}{4}}} \quad (4.5c)$$



which come from (2.11c) and (3.2c), we can rewrite (4.4c) as

$$u_i(\xi_i) = \frac{\frac{8\kappa^3}{a_i}(a_i^2 - 1)(a_i^2 - \frac{1}{4})}{\cosh \xi_i + 2a_i - \frac{1}{a_i}} \quad (4.6)$$

which is just (2.15a).

In the limit of  $t \rightarrow +\infty$ , on the other hand, the expressions corresponding to (4.4a) and (4.4b) turn out to be as

$$u \sim u_i(\xi_i + \gamma_i^{(-)}) \quad (4.7a)$$

$$x - c_i t - x_{i0} \sim \frac{\xi_i}{\kappa k_i} + \ln \left( \frac{1 + e^{\xi_i - \phi_i + \gamma_i^{(-)}}}{1 + e^{\xi_i + \phi_i + \gamma_i^{(-)}}} \right) - 2 \sum_{j=1}^{i-1} \phi_j + d \quad (4.7b)$$

where

$$\gamma_i^{(-)} = \sum_{j=i+1}^N \gamma_{ij}, (i = 1, 2, \dots, N). \quad (4.7c)$$

If one observes the behavior of the  $N$ -soliton solution described above in the coordinate system at rest, the asymptotic form of the solution as  $t \rightarrow \pm\infty$  is represented by a superposition of  $N$  single solitons

$$u \sim \sum_{i=1}^N u_i(\xi_i + \gamma_i^{(+)}, (t \rightarrow -\infty) \quad (4.8a)$$

$$u \sim \sum_{i=1}^N u_i(\xi_i + \gamma_i^{(-)}, (t \rightarrow +\infty). \quad (4.8b)$$

Note that if one shifts the phase  $\xi_i$  as  $\xi_i \rightarrow \xi_i - \sum_{\substack{j=1 \\ (j \neq i)}}^N \gamma_{ij}$ , then the argument of  $u_i$  in (4.8a) and (4.8b) becomes  $\xi_i - \gamma_i^{(-)}$  and  $\xi_i - \gamma_i^{(+)}$ , respectively. The resulting asymptotic expressions of  $u$  coincide with those derived in I by a different method (see formulas (4.45) and (4.50) of I). It should be emphasized, however, that these asymptotic formulas are valid in the  $(y, t)$  coordinate system. The new feature appears when we transform back to the original  $(x, t)$  coordinate system, which we shall now demonstrate. Indeed, as  $t \rightarrow -\infty$ , the center position of the  $i$ th soliton in the  $(y, t)$  coordinate is found from (4.8a) as  $\xi_i = -\gamma_i^{(+)}$ . It then follows from (4.4b) that the trajectory of the corresponding center position  $x_c$  in the  $(x, t)$  coordinate is described by the relation

$$x_c - c_i t - x_{i0} = -\frac{\gamma_i^{(+)}}{\kappa k_i} + \ln \left( \frac{1 + e^{-\phi_i}}{1 + e^{\phi_i}} \right) - 2 \sum_{j=1}^{i-1} \phi_j + d. \quad (4.9)$$

As  $t \rightarrow +\infty$ , the expression corresponding to (4.9) takes the form

$$x_c - c_i t - x_{i0} = -\frac{\gamma_i^{(-)}}{\kappa k_i} + \ln \left( \frac{1 + e^{-\phi_i}}{1 + e^{\phi_i}} \right) - 2 \sum_{j=i+1}^N \phi_j + d. \quad (4.10)$$

Let  $\Delta_i$  be the phase shift of the  $i$ th soliton which is defined by the shift of the trajectory of the  $i$ th soliton as  $t \rightarrow +\infty$  relative to its trajectory as  $t \rightarrow -\infty$ . This quantity is evaluated simply from (4.9) and (4.10), giving rise to the result

$$\Delta_i = \frac{1}{\kappa k_i} \left( \gamma_i^{(+)} - \gamma_i^{(-)} \right) + 2 \sum_{j=1}^{i-1} \phi_j - 2 \sum_{j=i+1}^N \phi_j, (i = 1, 2, \dots, N). \quad (4.11)$$

If we use the formula

$$e^{-\phi_i} = \frac{\alpha_i - 1}{\alpha_i + 1}, (i = 1, 2, \dots, N) \quad (4.12a)$$

with

$$\alpha_i = \sqrt{\frac{(2a_i - 1)(a_i + 1)}{(2a_i + 1)(a_i - 1)}}, (i = 1, 2, \dots, N) \quad (4.12b)$$

which is derived from (2.11c) and (3.2c), formula (4.11) reproduces the corresponding expression presented in I (see (4.52) of I). While the first term on the right-hand side of (4.11) is in accordance with the formula for the phase shift arising in the context of the shallow water wave equation [8], the second and third terms appear as a consequence of the coordinate transformation (2.3). In view of the latter terms, the characteristics of the interaction process of solitons are found to differ substantially from those derived from the shallow water theory. A full explanation about this topic has already been given in I in the case of the two-soliton solution.

## 5. Discussion

In conclusion, it will be worthwhile to discuss the structure of the  $N$ -soliton solution of the DP equation in conjunction with that of the following CH equation [19-21]

$$u_t + 2\kappa^2 u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \quad (5.1)$$

The multisoliton solutions of the CH equation with  $\kappa \neq 0$  have been constructed by several different methods [9, 22-27]. In particular, we have demonstrated by using an elementary theory of determinants that the  $N$ -soliton solution of the CH equation exhibits a parametric representation similar to the  $N$ -soliton solution (3.1) of the DP equation. Indeed, it can be written as [9]

$$u(y, t) = \left( \ln \frac{f_2}{f_1} \right)_t \quad (5.2a)$$

$$x(y, t) = \frac{y}{\kappa} + \ln \frac{f_2}{f_1} + d, \quad (5.2b)$$

where  $f_1$  and  $f_2$  are  $\tau$ -functions whose structure is essentially the same as that of the  $\tau$ -functions constructing the  $N$ -soliton solution of a model equation for shallow water waves introduced in the context of the inverse scattering transform (IST) method [26]. Thus, the difference between the  $\tau$ -functions  $g_1, g_2$  and  $f_1, f_2$  turns out to be the most important issue to be explored. Although  $f_1$  and  $f_2$  have the determinantal expressions [9],  $g_1$  and  $g_2$  are expressed by a finite sum as indicated in (3.2). An important observation is that  $g_1$  and  $g_2$  can be written in terms of pfaffians [12, 13, 29]. These  $\tau$ -functions satisfy the bilinear equation [8] for a model shallow water wave equation of Hirota and Satsuma which is a member of the BKP hierarchy. On the other hand, it is well known that  $f_1$  and  $f_2$  satisfy the bilinear equation [8, 25] for a model shallow water wave equation of Ablowitz et al which is a member of the AKP hierarchy. These inspections show clearly that the DP and CH equations belong to the different class of soliton equations. The discussion given here becomes more transparent if one compares the Lax pairs associated with both equations. Indeed, the spectral problem associated with the DP equation is described by the third order equation [2] while the one corresponding to the CH equation is the second order equation [20, 21]. As far as we know, however, the former spectral problem has not been studied sufficiently as yet. In this respect, it is an interesting problem to derive the  $N$ -soliton solution of the DP equation by means of IST.

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### Appendix. The CKP $\tau$ -function

We introduce the KP  $\tau$ -function in the standard Hirota form [30]

$$\tau_{2N} = \sum_{\mu=0,1} \exp \left[ \sum_{i=1}^{2N} \mu_i \tilde{\xi}_i + \sum_{\substack{i,j=1 \\ (i < j)}}^{2N} \mu_i \mu_j \tilde{A}_{ij} \right] \quad (A.1)$$

with

$$\tilde{\xi}_i = (\tilde{p}_i - \tilde{q}_i)y + \left( \frac{1}{\tilde{p}_i} - \frac{1}{\tilde{q}_i} \right) \kappa^2 t + \tilde{\xi}_{i0}, \quad (i = 1, 2, \dots, 2N) \quad (A.2)$$

$$e^{\tilde{A}_{ij}} = \frac{(\tilde{p}_i - \tilde{p}_j)(\tilde{q}_i - \tilde{q}_j)}{(\tilde{p}_i - \tilde{q}_j)(\tilde{q}_i - \tilde{p}_j)}, \quad (i, j = 1, 2, \dots, 2N; i \neq j) \quad (A.3)$$

where the notation  $\sum_{\mu=0,1}$  implies the summation over all possible combinations of  $\mu_1 = 0, 1, \mu_2 = 0, 1, \dots, \mu_{2N} = 0, 1$ . While the usual definition of the  $\tau$ -function introduces

an infinite number of independent variables, only the two variables  $y$  and  $t$  are retained and others are set to zero in (A.2). The CKP reduction is defined by the following parameterization [15]

$$\tilde{p}_{2i-1} = q_i, \tilde{p}_{2i} = p_i, \tilde{q}_{2i-1} = -p_i, \tilde{q}_{2i} = -q_i, (i = 1, 2, \dots, N). \quad (A.4)$$

In order to obtain the  $N$ -soliton solution of equation (2.9), the parameters  $p_i$  and  $q_i$  are specified by (2.11f) and (2.11g), respectively. It then follows from (A.1), (A.4), (2.11f) and (2.11g) that

$$\tilde{\xi}_{2i-1} = \tilde{\xi}_{2i} = \xi_i, (i = 1, 2, \dots, N). \quad (A.5)$$

Here,  $\xi_i$  is given by (2.11b). Note that the phase factors  $\tilde{\xi}_{i0}$  ( $i = 1, 2, \dots, 2N$ ) have been rewritten as  $\tilde{\xi}_{2i-1,0} = \tilde{\xi}_{2i,0} = (p_i + q_i)y_{i0}$  ( $i = 1, 2, \dots, N$ ). We denote the CKP  $\tau$ -function constructed above by  $\tau_{2N}|_{CKP}$ . We have already shown in I that the function  $v \equiv (3/4)(\ln f)_{yy}$  with  $f = \tau_{2N}|_{CKP}$  (see (2.10)) satisfies equation (2.9). We shift the phase of the  $i$ th soliton as  $\xi_i \rightarrow \xi_i - \ln a_i$  (or equivalently,  $y_{i0} \rightarrow y_{i0} + (\ln a_i/k_i)$ ) and rewrite  $f$  in the form

$$\begin{aligned} f = & \sum_{\mu=0,1} \exp \left[ \sum_{i=1}^N (\mu_{2i-1} + \mu_{2i})(\xi_i - \ln a_i) \right. \\ & + \frac{1}{2} \sum_{\substack{i,j=1 \\ (i \neq j)}}^N (\mu_{2i-1}\mu_{2j-1}A_{2i-1,2j-1} + \mu_{2i}\mu_{2j}A_{2i,2j}) \\ & + \frac{1}{2} \sum_{\substack{i,j=1 \\ (i \neq j)}}^N (\mu_{2i-1}\mu_{2j}A_{2i-1,2j} + \mu_{2i}\mu_{2j-1}A_{2i,2j-1}) \\ & \left. + \frac{1}{2} \sum_{i=1}^N (\mu_{2i-1}\mu_{2i}A_{2i-1,2i} + \mu_{2i}\mu_{2i-1}A_{2i,2i-1}) \right] \end{aligned} \quad (A.6)$$

where  $A_{2i-1,2j-1} (= A_{2i,2j})$  and  $A_{2i-1,2j} (= A_{2i,2j-1})$  are given respectively by (2.11d) and (2.11e). We replace the dummy indices  $\mu_i$  ( $i = 1, 2, \dots, 2N$ ) in accordance with the rule  $\mu_{2i-1} \rightarrow \mu_i, \mu_{2i} \rightarrow \nu_i$  ( $i = 1, 2, \dots, N$ ). If we substitute the relations  $A_{2i-1,2i} = A_{2i,2i-1} = 2 \ln a_i$  which can be derived from (2.11e), (2.11f) and (2.11g), then we arrive at formula (2.11a).

In the present analysis, we have obtained the  $\tau$ -function in the form of finite sum. We can also express it in terms of a determinant, as already demonstrated in I. To see the relationship between two alternative expressions of  $f$ , we use formula (3.7) with the phase variables  $\xi_i$  shifted appropriately. In the two-soliton case, for instance, we shift  $\xi_i$  as  $\xi_i \rightarrow \xi_i - \ln \delta$  ( $i = 1, 2$ ) in (2.12b) and then multiply it by a constant factor  $(a_1 a_2 \delta)^2$ . We can confirm that the resultant expression coincides with the corresponding  $\tau$ -function presented in I (see (4.18) of I).

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